Localization for random perturbations of periodic Schrödinger operators with regular Floquet eigenvalues

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Abstract

We prove a localization theorem for continuous ergodic Schrödinger operators $H_\omega := H_0 + V_\omega$, where the random potential V_ω is a nonnegative Anderson-type perturbation of the periodic operator H_0. We consider a lower spectral band edge of $\sigma(H_0)$, say $E = 0$, at a gap which is preserved by the perturbation V_ω. Assuming that all Floquet eigenvalues of H_0, which reach the spectral edge 0 as a minimum, have there a positive definite Hessian, we conclude that there exists an interval I containing 0 such that H_ω has only pure point spectrum in I for almost all ω.

1 Introduction and results

Localization

Already in the fifties Anderson [1] concluded by physical reasoning that some random quantum Hamiltonians on a lattice should exhibit *localization* in certain energy regions. That is to say that the corresponding self-adjoint operator has pure point spectrum in these energy intervals.

Since then mathematical physicists developed a machinery to prove rigorously this phenomenon from solid state physics. Most of them used the so-called *multi scale analysis* (MSA) introduced in a paper by Fröhlich and Spencer [14] to prove a weaker form of localization at low energies for the discrete analogue of the Schrödinger operator. This quite complicated reasoning was streamlined by von Dreifus and Klein [44]. The underlying lattice structure made the MSA easier to apply to discrete Hamiltonians but soon adaptations for continuous

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Schrödinger operators followed [29, 23, 6, 24]. We prove in Theorem 1.1 a localization result for energies near internal spectral edges of a periodic Schrödinger operator H_0 which is perturbed by an Anderson-type potential V_ω. Unlike [2, 21] our results are not restricted to a special disorder regime of the random coupling constants in V_ω. Instead we assume that the periodic operator H_0 has regular Floquet eigenvalues. This behaviour is commonly assumed among physicists. Recent results by Klopp and Ralston indicate that it is generic [27].

In the remainder of this section we introduce our model, state the main Theorem 1.1 and the technical Proposition 1.2 on which it is based. Section 2 explains how to deduce Theorem 1.1 from Proposition 1.2, in Section 3 we describe the functional calculus with almost analytic functions, Section 4 contains a comparison result between the integrated density of states on finite cubes and on the whole of $\mathbb{R}^d$ and the last Section 5 deals with periodic (or more generally quasi-periodic) boundary conditions which are necessary to complete the proof of Proposition 1.2. Two technical proofs are placed in an appendix.

The model

On the Hilbert space $L^2(\mathbb{R}^d)$ we consider a self-adjoint operator $H := H_\omega$ made up of a periodic Schrödinger operator H_0 and a random perturbation V_ω

$$H_\omega := H_0 + V_\omega \ . \tag{1}$$

Here $H_0 := -\Delta + V_0$ is the sum of the negative Laplacian and a $\mathbb{Z}^d$-periodic potential $V_0 \in L^p_{loc}(\mathbb{R}^d)$ with $p = 2$ if $d \leq 3$, $p > 2$ if $d = 4$ and $p \geq d/2$ if $d \geq 5$. Such a potential is an infinitesimal perturbation of $-\Delta$ so the sum is self-adjoint with domain $D(-\Delta) = W^2_2(\mathbb{R}^d)$, the Sobolev space of L^2-functions whose second derivative is also in L^2 (cf. [34, 35]). The random perturbation is of *Anderson type*

$$V_\omega(x) := \sum_{k\in\mathbb{Z}^d} \omega_k \, u(x-k) \ , \tag{2}$$

where $(\omega_k)_{k\in\mathbb{Z}^d}$ is a collection of independent identically distributed (i.i.d.) random variables, called *coupling constants*. Their distribution has a bounded density with support $[0, \omega_{\max}]$ for some $\omega_{\max} > 0$. The non-negative *single site potential* u has to decay exponentially and have an uniform lower bound on some open subset of $\mathbb{R}^d$, more precisely

$$u \geq \delta_1 \chi_\Lambda, \ \delta_1 > 0 \text{ where } \Lambda := \Lambda_s := \{x \in \mathbb{R}^d | \ \|x\|_\infty < s/2\}, \ s > 0$$

and

$$\|\chi_{\Lambda_1} u(\cdot - l)\|_{L^p} \leq \delta_2 \, e^{-\delta_3 l}, \ \delta_2, \delta_3 > 0 \ . \tag{3}$$

H_ω is an ergodic operator and we infer from [19, 4] or [33] that there exists a set $\sigma \subset \mathbb{R}$ such that $\sigma = \sigma(H_\omega)$ for almost all $\omega \in \Omega$, i.e. the spectrum of H_ω is almost surely non-random. In the same sense σ_{ac}, σ_{sc} and σ_{pp} are ω-independent subsets of the real line.

Under some mild assumptions the periodic background operator H_0 has a spectrum with *band structure*, i.e. $\sigma(H_0) = \bigcup_{n\in\mathbb{N}} [E_n^-, E_n^+]\,, \ E_1^- \leq E_1^+ \leq$

$E_2^- \leq \ldots$, where for some n we have open spectral gaps, i.e. $E_n^+ < E_{n+1}^-$ (cf. [9, 39, 35]). We assume that there exist positive numbers a, b and b' with

$$[0,a] \subset \sigma(H_0),\ \ [-b,0[\subset \rho(H_0) \text{ and } [-b',0[\subset \rho(H_\omega)\,.$$

Since 0 is in the support of the density of ω_0 it follows that $0 \in \sigma(H_\omega)$. In this case we say that 0 is a *lower band edge* of the periodic operator, which is preserved by the positive random perturbation V_ω.

H_0 can be decomposed into a *direct integral* via an unitary transformation U (cf. [39, 35])

$$U H_0\, U^* = \int_{[-\pi,\pi]^d}^{\oplus} H_0|_{\Lambda_1}^{\theta}\, d\theta \ . \tag{4}$$

Here $H_0|_{\Lambda_1}^{\theta}$ is the same formal differential expression as H_0 acting on functions $f \in W_2^2(\Lambda_1)$ with *θ-boundary conditions*, i.e. for all $j = 1,\ldots,d$ we have a phase shift in the corresponding direction: $f(x+e_j) = e^{i\theta_j} f(x)$ where $x_j = -1/2$. It is an operator with discrete spectrum, which consists of the so-called *Floquet eigenvalues*

$$E_1(\theta) \leq \ldots \leq E_n(\theta) \leq \ldots \qquad n \in \mathbb{N}\ .$$

These are Lipschitz-continuous on $[-\pi,\pi]^d$. In fact they "generate" the bands of the spectrum of H_0

$$\sigma(H_0) = \bigcup_{n\in\mathbb{N}} \bigcup_{\theta\in[-\pi,\pi]^d} E_n(\theta)\ .$$

There is a finite set of indices $\mathcal{N} \subset \mathbb{N}$ (cf. [39]) such that

$$E_n(\theta) = 0 \text{ for some } \theta \in [-\pi,\pi]^d \Longrightarrow\ n \in \mathcal{N}\ .$$

Since 0 is a lower band edge of $\sigma(H_\omega)$, $E_n(\theta) = 0$ has to be a minimum of $E_n(\cdot)$. If for all $n \in \mathcal{N}$, $E_n(\cdot)$ has only quadratic minima at 0 (i.e. the Hessian of $E_n(\cdot)$ at any minimum with value 0 is positive definite) we say that H_0 has *regular Floquet eigenvalues at* 0.

Results

Our result on localization at an lower internal spectral band edge is the following

Theorem 1.1 *If H_0 has regular Floquet eigenvalues at* 0 *and H_ω is constructed as above, then there exists a number $E_0 > 0$ such that*

$$[0,E_0] \subset \sigma_{pp}(H_\omega),\ \ [0,E_0] \cap \sigma_c(H_\omega) = \emptyset\ .$$

The proof of the theorem is based on the following proposition.

Proposition 1.2 *Assume that H_0 has regular Floquet eigenvalues at 0 and H_ω is constructed as above. Then for all $q > 0$ and $\alpha \in]0, 1[$ there exists a $l_0 := l_0(q, \alpha) \in \mathbb{N}$ such that for all $l \geq l_0$ we have*

$$\mathbb{P}\{\omega | \ \sigma(H_\omega|_{\Lambda_l}^{per}) \cap [0, l^{-\alpha}[\neq \emptyset\} \leq l^{-q}.$$

Here the index "per" denotes periodic boundary conditions on the cube Λ_l.

The statement of Proposition 1.2 remains true if we replace the periodic boundary conditions by general θ-boundary conditions with $\theta \in [\frac{-\pi}{2l+1}, \frac{\pi}{2l+1}]^d$, cf. (4) and (27). The proof of the proposition is given in sections 3 to 5. It uses the existence of *Lifshitz-tails* of the *integrated density of states* (IDS) of the ergodic operator H_ω if H_0 has regular Floquet eigenvalues, which was proved by Klopp in [25], who also noted that his result could be used for a localization proof.

Theorem 1.1 is proved using the MSA. Since this technique is well understood by now [6, 21, 41] we only sketch it to show how Proposition 1.2, which is the main technical novelty of this paper, enters. This is done in Section 2, where also a discussion of previous results can be found.

Remark 1.3 At any lower band edge one can prove localization under the analogous assumptions. Here $E = 0$ was chosen only for notational simplicity. If the Anderson-type perturbation V_ω is negative our theorem can be used to establish localization on any upper band edge with regular Floquet eigenvalues.

If the underlying $\mathbb{Z}^d$ is replaced by some other Euclidean lattice

$$\Gamma := \{\gamma \in \mathbb{R}^d | \ \gamma = \sum_{j=1}^{d} \beta_j a_j, \beta \in \mathbb{Z}^d\} \ ,$$

where $\{a_j\}_{j=1}^d$ is a basis of $\mathbb{R}^d$, the same theorem and proposition are valid by a simple modification of the proofs.

In any case we will use the maximum norm when considering lattice points k or γ in $\mathbb{Z}^d$ or Γ, i.e. $|\gamma| := \|\gamma\|_\infty := \max\{|\gamma_j|, \ j = 1, \dots, d\}$, where $(\gamma_1, \dots, \gamma_d) \in \mathbb{R}^d$ are the components of γ.

An inspection of our proofs and the papers [25, 26] and [21, 46] shows that Proposition 1.2 and Theorem 1.1 extend to single site potentials u with sufficiently fast polynomial decay (in L^p-sense), cf. (12).

Example 1.4 Finally we give an example of a periodic operator which has only regular Floquet eigenvalues at all band edges. Thus we know that our condition in the above theorem is fulfilled and we can prove localization at any lower band edge. Let V_0 satisfy the conditions posed above on the periodic potential and let it be a sum of potentials V_j which are periodic in the jth coordinate direction and constant in all the others; more precisely

$$V_0(x) := \sum_{j=1}^{d} V_j(x_j)$$

where $V_j : \mathbb{R} \to \mathbb{R}$ is a periodic function and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. Then both H_0 and $H_0|_{\Lambda_1}^{\theta}$ can be decomposed into a direct sum of one-dimensional

operators. For these it is known that all Floquet eigenvalues are regular [9, 25]. As the eigenvalues of the direct sum are just sums of the eigenvalues of the one-dimensional operators it is clear that the former also have to be regular.

Corollary 1.5 *Let the ergodic operator $H_\omega := -\Delta + V_0 + V_\omega$ be constructed as above and the periodic potential be decomposable, i.e.*

$$V_0(x) := \sum_{j=1}^{d} V_j(x_j) \ .$$

Let E be a lower spectral band edge of the periodic operator $H_0 := -\Delta + V_0$ at a spectral gap which is not closed by the perturbation V_ω. Then there exists an interval $I \ni E$ such that

$$I \subset \sigma_{pp}(H_\omega), \ \ \sigma_c(H_\omega) \cap I = \emptyset.$$

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2 Multi scale analysis and associated ideas

In this section we explain how Theorem 1.1 is deduced from Proposition 1.2 and discuss previous localization results.

An intermediary step in the proof of localization is the establishing of the exponential decay of the resolvent

$$\sup_{\epsilon \neq 0} \|\chi_x R(\epsilon) \chi_y\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq const\, e^{-c|x-y|} \text{ for almost all } \omega \ , \tag{5}$$

where $R := R(\epsilon) := (H_\omega - E - i\epsilon)^{-1}$ is the resolvent of H_ω near an energy value E in the energy interval $I \in \mathbb{R}$ for which we want to prove localization. The χ_x and χ_y are characteristic functions of unit cubes centered at x, respectively at y. This bound can be used to rule out absolutely continuous spectrum [30] and is interpreted as absence of diffusion [14, 29] in the energy region I if (5) holds for all $E \in I$.

It turns out that the finite size resolvent $R_\Lambda(\epsilon) := (H_\omega|_\Lambda - E - i\epsilon)^{-1}$ is easier approachable than $R(\epsilon)$ on the whole space. Here $H_\omega|_\Lambda$ is the restriction of H_ω to $L^2(\Lambda)$ with some appropriate boundary conditions (b.c.); the use of Dirichlet or periodic b.c. is most common. However the operator $H_\omega|_\Lambda$ is not ergodic and for its resolvent an estimate like (5) can be expected to hold only with a probability strictly smaller than one. This is the place where MSA enters. It is an induction argument over increasing length scales l_j. They are defined recursively by $l_{j+1} := [l_j^\zeta]_3$, where $[l_j^\zeta]_3$ is the greatest multiple of 3 smaller than l_j^ζ. The scaling exponent ζ has to be from the interval $]1, 2[$. On each scale one

considers the box resolvent $R_j(\epsilon) := R_{\Lambda_j}(\epsilon)$ and proves its exponential decay with a probability which tends to 1 as $j \to \infty$. We outline briefly the ingredients of the MSA as it is given in [6, 21] or [4].

First we explain some notation which is used afterwards. Let $\delta > 0$ be a small constant independent of the length scale l_j and $\phi_j(x) \in C^2$ a function which is identically equal to 0 for x with $\|x\|_\infty > l_j - \delta$ and identically equal to one for x with $\|x\|_\infty < l_j - 2\delta$. The commutator $W(\phi_j) := [-\Delta, \phi_j] := -(\Delta\phi_j) - 2(\nabla\phi_j)\nabla$ is a local operator acting on functions which live on a ring of width δ near the boundary of $\Lambda_j := \Lambda_{l_j}$. We say that a pair $(\omega, \Lambda_j) \in \Omega \times \mathcal{B}(\mathbb{R}^d)$ is *m-regular*, if

$$\sup_{\epsilon \neq 0} \|W(\phi_j) R_j(\epsilon) \chi_{l_j/3}\|_{\mathcal{L}} \leq e^{-ml_j} \; . \tag{6}$$

Here $\|\cdot\|_{\mathcal{L}}$ is the operator norm on $L^2(\Lambda_j)$ and $\chi_{l_j/3}$ the characteristic function of $\Lambda_{l_j/3} := \{y| \, \|y\|_\infty \leq l_j/6\}$. Thus the distance of the supports of $\nabla\phi_j$ and $\chi_{l/3}$ is at least $l_j/3 - 2\delta \geq l_j/4$.

Let $q_0 > 0$ and $m_0 \geq const\, l_0^{-1/4}$. The starting point of the MSA is the estimate

$$\text{(H1)}(l_0, m_0, q_0) \qquad\qquad \mathbb{P}\{\omega| \, (\omega, \Lambda_0) \text{ is } m_0\text{-regular}\} \geq 1 - l_0^{q_0}$$

which serves as the base clause of the induction. The induction step consists in proving

$$\text{(H1)}(l_j, m_j, q_j) \Longrightarrow \text{(H1)}(l_{j+1}, m_{j+1}, q_{j+1}) \tag{7}$$

For the mass of decay m_{j+1} and the probability exponent q_{j+1} on the scale l_{j+1} the following estimates are valid

$$\forall \xi > 0 \; \exists c_1, c_2, c_3 \text{ independent of } j \text{ such that}$$

$$m_{j+1} \;\geq\; m_j \left(1 - \frac{4l_j}{l_{j+1}}\right) - \frac{c_1}{l_j} - c_2 \frac{\log l_{j+1}}{l_{j+1}} \tag{8}$$

$$l_{j+1}^{q_{j+1}} \;\leq\; c_3 \left(\frac{l_{j+1}}{l_j}\right)^{2d} l_j^{2q_j} + \frac{1}{2} l_{j+1}^{-\xi} \; . \tag{9}$$

For the recursion clause (7) a *Wegner estimate* [45] is needed:

$$\text{(H2)} \qquad\qquad \mathbb{P}\{\omega| \, d(\sigma(H_\omega|_\Lambda), E) \leq \eta\} \leq C_W \, \eta |\Lambda|^2$$

for all boxes $\Lambda \subset \mathbb{R}^d$ and all $\eta > 0$, such that $[E - \eta, E + \eta]$ is contained in a suitable small energy interval near the spectral band edge (cf. Theorem 3.1 in [21]). Here $|\Lambda|$ stands for the Lebesgue measure of the cube Λ.

The deterministic part of the induction step uses the *geometric resolvent formula* [6, 17]

$$\phi_\Lambda (H_{\Lambda'} - z)^{-1} = (H_\Lambda - z)^{-1} \phi_\Lambda + (H_\Lambda - z)^{-1} W(\phi_\Lambda)(H_{\Lambda'} - z)^{-1} \tag{10}$$

for $z \in \rho(H_{\Lambda'}) \cap \rho(H_\Lambda)$ and $\phi_\Lambda \in C^2$ with support in $\Lambda \subset \Lambda'$. It gives the estimate

$$\|\chi_{l/3}(\cdot - x) R_{3l'}(\epsilon) \chi_{l/3}(\cdot - y)\|_{\mathcal{L}} \leq (3^d e^{-ml})^{3|x-y|l^{-1}-4} \|R_{3l'}(\epsilon)\|_{\mathcal{L}} \tag{11}$$

if no two disjoint non-regular boxes $\Lambda_l \subset \Lambda_{l'}$ with center in $\frac{l}{3}\mathbb{Z}^d \cap \Lambda_{3l'}$ exist for ω. In our case $l := l_j$ is the length scale on which the exponential decay of the resolvent is already known and $l' := l_{j+1}$ the scale on which we want to prove it. By the estimates (H1),(H2) we have with probability $1 - l_{j+1}^{q_{j+1}}$ (bounded by the inequality (9)) exponential decay on the length scale l_{j+1} with mass m_{j+1} (bounded as in (8)).

We stated above the ingredients of the MSA as they are valid if u is supported in Λ_1. If the single site potential is of long range type as in (3) one has to use the adapted MSA from the papers [21, 46].

Once the estimate (H1) is established on all length scales $l_j, j \in \mathbb{N}$, one infers an exponential decay estimate for the resolvent on the whole of $\mathbb{R}^d$. Afterwards one uses a spectral averaging technique (cf.[6]) based on ideas of Kotani, Simon, Wolf and Howland to conclude localization [28, 38, 18]. An alternative version of the MSA can be found in the recently published book [41].

Recent papers concentrate on proofs for the Wegner estimate and the initial length scale decay of the resolvent. At the same time adaptations of the MSA for various random Schrödinger operators, as well as Hamiltonians governing the motion in classical physics appeared [10, 11, 7, 40]. Recently Najar [32] obtained analog results to [25] and the present paper concerning Lifshitz tails and localization for acoustic operators.

We discuss briefly some results for quantum mechanical Hamiltonians.

In [24] Klopp proved a Wegner lemma for energies at the infimum of the spectrum which applies to an Anderson perturbation V_ω with single site potentials u that are allowed to change sign, cf. also [43, 16]. For V_ω a Gaussian random field a Wegner estimate was shown in [12]. Its main feature is that no underlying lattice structure of V_ω is needed. This result allows one to conclude localization for the corresponding Schrödinger operator at low energies [13]. Kirsch, Stollmann and Stolz proved in [21] (cf. also [46]) a Wegner estimate with only polynomial decay conditions on the single site potential u and deduced a localization result for Hamiltonians with long range interactions. They require

$$|u(x)| \leq const\,(|x| + 1)^{-m} \text{ for some } m > 4d\ . \tag{12}$$

The resolvent decay estimate (H1) for some initial length scale can be proved with semiclassical techniques. Using the Agmon metric one can achieve rigorously decay bounds with what is called among physicists WKB-method [6, 17]. However this reasoning is only applicable for energies near the bottom of the spectrum.

The so-called *Combes-Thomas argument* [5] allows one to infer the following inequality

$$\|\chi_x (H - z)^{-1} \chi_y\|_{\mathcal{L}} \leq const\, d(\sigma(H), z)^{-1}\, e^{-const\, d(\sigma(H),z)\, |x-y|} \tag{13}$$

where H is a self-adjoint Schrödinger operator on $L^2(\mathbb{R}^d)$ and $z \in \rho(H)$. It was first applied to multiparticle Hamiltonians [5], but it is also useful in our case, as soon as we get a lower bound on $d(\sigma(H_\omega|_\Lambda), z)$. Thus it is sufficient to prove an estimate like

$$\mathbb{P}\{\omega |\ d(\sigma(H_\omega|_{\Lambda_l}, I) < l^{-\alpha}/2\ \} \leq l^{-q} \tag{14}$$

for some $\alpha \in]0, 1/4]$. Such a bound follows immediately from Proposition 1.2 with $I := [0, \frac{1}{2}\, l^{-\alpha}[$, for $l > (2b')^{-1/\alpha}$. Now Inequality (13) implies the initial scale estimate (H1) with $m_0 \geq const\, l^{-1/4}$ for l large and $E \in I$, cf. [21, Lemma 5.5]. The constant depends on the energy and the potential, but not on l and m_0.

Two possibilities were used to deduce (14). The first is to assume a special disorder regime, more precisely to demand a sufficiently fast decay of the density g of the distribution of ω near the endpoints 0 and $\omega_{\max}$ of $\operatorname{supp} g$:

$$\exists \tau > d/2 \ : \ \forall \text{ small } \epsilon > 0$$
$$\int_0^\epsilon g(s)ds \leq \epsilon^\tau, \text{ respectively } \int_{\omega_{\max}-\epsilon}^{\omega_{\max}} g(s)ds \leq \epsilon^\tau$$

depending on whether one wants to consider a lower or upper band edge. This approach was used in [6, 21]. Its shortcoming is that it excludes quite a few distributions, e.g. the uniform distribution on $[0, \omega_{\max}]$.

The other way to prove (14), which we pursue, is to use the existence of Lifshitz tails of the integrated density of states at the edges of the spectrum. One defines the IDS usually as follows:

$$N(E) \;:=\; \lim_{\Lambda \nearrow \mathbb{R}^d} N(H_\omega|_\Lambda^D, E) \tag{15}$$

$$:=\; \lim_{\Lambda \nearrow \mathbb{R}^d} |\Lambda|^{-1} \#\{ \text{ eigenvalues of } H_\omega|_\Lambda^D \text{ below } E\} \ , \tag{16}$$

i.e. as the limit of the normalized counting function of eigenvalues of a box Hamiltonian. Here $H_\omega|_\Lambda^D$ is the restriction of H_ω to $L^2(\Lambda)$ with Dirichlet b.c. As $H_\omega|_\Lambda^D$ has compact resolvent and hence discrete spectrum, definition (15) makes sense. $N(E)$ is almost surely ω-independent and the use of Dirichlet b.c. in its definition implies [20]

$$N(E) = \sup_{\Lambda \nearrow \mathbb{R}^d} N(H_\omega|_\Lambda^D, E) \ . \tag{17}$$

One says that $N(\cdot)$ exhibits Lifshitz tails at some spectral edge $\mathcal{E}$ if

$$\lim_{E \to \mathcal{E}} \frac{\log|\log|N(E) - N(\mathcal{E})||}{\log|E - \mathcal{E}|} = -\frac{d}{2} \ . \tag{18}$$

At the infimum of the spectrum, i.e. for $\mathcal{E} = \inf \sigma(H_\omega)$, (17) and (18) imply

$$\#\{\text{eigenvalues of } H_\omega|_\Lambda^D \text{ in } [\mathcal{E}, E]\} \leq |\Lambda| N(E) \leq |\Lambda| \exp(-cE^{-d/4})$$

since $N(\mathcal{E}) = 0$. This estimate was used in [24] together with a Čebišev inequality to prove (H1) at the bottom of the spectrum, see also [29].

If one considers an internal band edge $\mathcal{E}$, Lifshitz asymptotics are not so easy to exploit since (17) cannot be directly used to bound

$$|N(H_\omega|_\Lambda, \mathcal{E}) - N(H_\omega|_\Lambda, E)| \ .$$

Therefore a comparison technique between $N(\cdot)$ and $N(H_\omega|_\Lambda, \cdot)$ is needed. In the one-dimensional case Mezincescu [31] proved Lifshitz tails at internal band

edges as well as a comparison lemma for the IDS (Lemma 2, in Section 4). This proof relies on the delicate analysis of Dirichlet eigenfunctions of $H_\omega|_\Lambda$ and their roots. The results in [31] make a localization proof in the one-dimensional case possible [42].

We prove in Section 4 an approximation result (Theorem 4.1) for the IDS of the multi-dimensional operator H_ω, which enables us to prove Proposition 1.2. In our case however periodic b.c. seem to be more efficient than Dirichlet b.c. since H_ω is a perturbation of a periodic operator.

In [25] it was proved that the IDS of H_ω exhibits Lifshitz asymptotics at a lower band edge $\mathcal{E}$ if before the perturbation V_ω the Floquet eigenvalues of the periodic background operator H_0 at $\mathcal{E}$ were regular. Thus our approximation theorem can be applied to conclude localization.

3 The Helffer-Sjöstrand formula: Functional calculus with almost analytic functions

In this section we introduce the Helffer-Sjöstrand formula (19) which is exploited in Section 4 to prove the IDS approximation result.

For an self-adjoint operator on $L^2(\mathbb{R}^d)$ and a complex-valued measurable function $f : \mathbb{R} \to \mathbb{C}$ one can define the operator

$$f(A) \text{ with domain } D(f(A)) := \{\psi \in L^2(\mathbb{R}^d) |\ f(A)\psi \in L^2(\mathbb{R}^d)\}$$

via the spectral theorem. The latter is normally proved using Riesz' representation theorem for $C(K)^*$, where K is a compact metric space, and the Cayley-transform if A is unbounded. Helffer and Sjöstrand [15] proved the following representation formula

$$f(A) := \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}}(z)(z - A)^{-1} dz \wedge d\bar{z} \tag{19}$$

if f is smooth and compactly supported. Here $\tilde{f} : \mathbb{C} \to \mathbb{C}$ denotes an *almost analytic extension* of $f : \mathbb{R} \to \mathbb{C}$. Davies [8] uses equation (19) as a starting point to develop systematically a functional calculus equivalent to the standard one. For further details on the material of this section see his book.

Definition 3.1 *For $n \in \mathbb{N}$ and $f \in C_0^n(\mathbb{R}, \mathbb{C})$ define the almost analytic extension (of order n) $\tilde{f} : \mathbb{C} \to \mathbb{C}$ by*

$$\tilde{f}(x, y) := \tilde{f}_n(x, y) := \left(\sum_{r=0}^{n} f^{(r)}(x) \frac{(iy)^r}{r!} \right) s(x, y)\ , \tag{20}$$

where we used the convention $z := x + iy := (x, y) \in \mathbb{C}$. The cutoff function s is defined with the abbreviation $\langle x \rangle := \sqrt{x^2 + 1}$ by the formula

$$s(x, y) := t\left(\frac{y}{\langle x \rangle}\right),\ t \in C_0^\infty(\mathbb{R})$$

with $t(x) = 0$ for $|x| > 2$, $t(x) = 1$ for $|x| < 1$ and $\|t'\|_\infty \leq 2$.

With this choice of the almost analytic extension formula (19) holds true. If the support of f is contained in $[-R,R]$, $\tilde{f}$ vanishes outside the set $\{z \in \mathbb{C} |\ x \in \operatorname{supp} f, |y| < 2R+2\}$. A calculation of the derivatives shows

$$\begin{aligned}
\frac{\partial \tilde{f}_n}{\partial \bar{z}}(z) &= \frac{1}{2}\left(\frac{\partial \tilde{f}_n}{\partial x} + i\frac{\partial \tilde{f}_n}{\partial y}\right)(z) \qquad (21)\\
&= \frac{1}{2} f^{(n+1)}(x)\frac{(iy)^n}{n!}\ s(x,y) + \frac{1}{2}(s_x(x,y) + is_y(x,y))\ \sum_{r=0}^{n} f^{(r)}(x)\frac{(iy)^r}{r!}\ .
\end{aligned}$$

By calculating the partial derivatives of s we see

$$|s_x + is_y| \leq \frac{6}{\langle x \rangle}\chi_{\{\langle x\rangle < |y| < 2\langle x \rangle\}}\ , \tag{22}$$

which shows that they vanish for $|y| \leq 1$ since always $\langle x \rangle = \sqrt{x^2+1} \geq 1$. Putting the bounds together we get

$$\left|\frac{\partial \tilde{f}_n}{\partial \bar{z}}(x,y)\right| \leq \frac{1}{2n!}|f^{(n+1)}s|\ |y|^n + \frac{3}{\langle x \rangle}\chi_{\{\langle x\rangle < |y| < 2\langle x \rangle\}} \sum_{r=0}^{n} |f^{(r)}|\frac{|y|^r}{r!}\ . \tag{23}$$

Later on f will be an approximation of the characteristic function $\chi_{[0,E]}$. It is going to have support inside $[-E/2, 2E]$ and be equal to 1 on $[0,E]$. One can choose f in such a way that $\|f^{(n)}\|_\infty \leq CE^{-n}$ and

$$|||f|||_n := \sum_{i=1}^{n} \|f^{(n)}\|_\infty \leq \tilde{C}E^{-n} \tag{24}$$

for sufficiently small E. The constants $C, \tilde{C}$ are independent of E.

4 IDS approximation theorem

In this section we bound the difference of the IDS of the ergodic operator H_ω and its periodic approximation $H_{\omega,l}$ which will be defined shortly. The estimate is contained in Theorem 4.1 which is the main technical result of this paper. Furthermore, it enables us to show in Theorem 4.6 that the IDS of the periodic approximation $H_{\omega,l}$ exhibits a kind of Lifshitz tail, if the IDS N of the original operator H_ω does so. The periodic approximation $H_{\omega,l}$ is defined by

$$H_{\omega,l}(x) := H_0(x) + \sum_{k\in\mathbb{Z}^d} \omega_{\tilde{k}}\, u(x-k) \tag{25}$$

where $\tilde{k} := k\ (\mathrm{mod}(2l{+}1)\mathbb{Z}^d)$. For any $l \in \mathbb{N}$ and $\omega \in \Omega$ it is a (2l+1)$\mathbb{Z}^d$-periodic operator. Our assumptions on u and ω ensure that it is an infinitesimally small perturbation of H_0, uniformly in l and ω. Hence it is a lower bounded symmetric operator which is self-adjoint on the domain $W_2^2(\mathbb{R}^d)$. Its IDS is defined by (cf. [35, 36, 39])

$$N_{\omega,l}(E) := N(H_{\omega,l}, E) := (2\pi)^{-d} \sum_{n\in\mathbb{N}} \int_{B_l} \chi_{\{E_n(\theta)<E\}}\, d\theta\ . \tag{26}$$

Here $E \in \mathbb{R}$ is an energy value, $E_n(\theta)$ is the n-th eigenvalue of $H_{\omega,l}|^{\theta}_{\Lambda_{2l+1}}$ and

$$\theta \in B_l := \left[\frac{-\pi}{2l+1}, \frac{\pi}{2l+1}\right]^d \tag{27}$$

if H_ω is $\mathbb{Z}^d$-ergodic. For some other Euclidean lattice it has to be replaced by the basic cell of the corresponding dual lattice $\Gamma^* := \{\gamma^* \in (\mathbb{R}^d)^* = \mathbb{R}^d | \forall \gamma \in \Gamma : \gamma^* \cdot \gamma \in 2\pi\mathbb{Z}\}$. We prove the following approximation result:

Theorem 4.1 *Let H_ω be defined as in Section 1 and $H_{\omega,l}$ as above. Denote by N, respectively $N_{\omega,l}$ the corresponding IDS'. For a real valued function $g \in C_0^{n+1}$ with support in $[-1/2, 1/2]$ we have*

$$\left| \mathbb{E}\left(\int_{\mathbb{R}} g(x) dN_{\omega,l}(x)\right) - \int_{\mathbb{R}} g(x) dN(x) \right| \leq \text{const.} \ |\operatorname{supp} g| \ |||g|||_{n+1} \, l^{-n+2d+1}$$

for sufficiently large $l \in \mathbb{N}$.

The proof is split into several lemmata. Remark 4.2 and Lemma 4.3 are taken from Section 5.2 of [25]. We denote with χ_l the characteristic function of the periodicity cell $\Lambda_{2l+1} := \{x \in \mathbb{R}^d | \ \|x\|_\infty \leq l + 1/2\}$ of $H_{\omega,l}$ and by $\chi_{l,\gamma}(x) := \chi_l(x - \gamma)$ its translation by $\gamma \in \mathbb{Z}^d$.

Remark 4.2 Note that one can infer from [3, 4],[33] and [25] the following equalities

$$\int_{\mathbb{R}} g(x) \, dN(x) = \mathbb{E}\left(\operatorname{Tr} \chi_0 \, g(H_\omega) \chi_0\right) \tag{28}$$

respectively

$$\int_{\mathbb{R}} g(x) \, dN_{\omega,l}(x) = (2l+1)^{-d} (\operatorname{Tr} \chi_l \, g(H_{\omega,l}) \chi_l) \ . \tag{29}$$

Using the decomposition

$$\chi_l = \sum_{k \in \mathbb{Z}^d, |k| < 2l+1} \chi_{0,k} \ ,$$

the $(2l+1)\mathbb{Z}^d$-periodicity of $H_{\omega,l}$ and the i.i.d. property of $(\omega_k)_{k \in \mathbb{Z}^d}$ one gets

$$\mathbb{E}\left(\int_{\mathbb{R}} g(x) \, dN_{\omega,l}(x)\right) = \mathbb{E}\left(\operatorname{Tr} \chi_0 \, g(H_{\omega,l}) \chi_0\right) \ . \tag{30}$$

Since $H_{\omega,l}$ is uniformly lower bounded there exists a $\lambda \geq 0$ such that $\mathrm{Id} \leq \lambda + H_{\omega,l}$ and $\mathrm{Id} \leq \lambda + H_\omega$ for all l, ω. From [37] we know that the operator $\chi_l(\lambda + H_{\omega,l})^{-q}(z - H_{\omega,l})^{-1}$ is trace class for all $q > d/2$. Using results from the appendix of [22] we infer

$$\|\chi_{0,\beta}(z - H_\omega)^{-1}(\lambda + H_\omega)^{-q} \chi_0\|_{\operatorname{Tr}} \quad \leq \quad \frac{\tilde{C}_1}{|y|} \exp(-|y| \, |\beta| / \tilde{C}_1) \tag{31}$$

for some $\tilde{C}_1 \geq 1$ independent of ω. This estimate is in fact a sophisticated version of the Combes-Thomas argument which we encountered already in Section 2. A simple resolvent estimate gives

$$\|\chi_0(z-H_{\omega,l})^{-1}T_\gamma\, u\, \chi_{0,\beta+\gamma}\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq \frac{\tilde{C}_1}{|y|}\|\chi_{0,\beta+\gamma}u\|_{L^p}\ , \tag{32}$$

where T_γ is the translation by $\gamma \in \mathbb{Z}^d$. As the single site potential u decays exponentially, inequality (32) gives a exponential bound in $-|\gamma+\beta|$. If one assumes that u decays polynomially with a sufficiently negative exponent, one still can carry trough the proof of Theorem 4.1.

Lemma 4.3 *If $g \in C_0^{n+1}$ and $\tilde{f}$ is an almost analytic extension of $f(x) := (\lambda+x)^q g(x)$, one has*

$$\begin{aligned} &\left|\mathbb{E}\left(\int_{\mathbb{R}} g(x)\,dN_{\omega,l}(x)\right) - \int_{\mathbb{R}} g(x)\,dN(x)\right| \\ &\leq \frac{C_1}{2\pi}\int_{\mathbb{C}} |y|^{-2}\left|\frac{\partial\tilde{f}}{\partial\bar{z}}(x,y)\right|\left(\sum_{\substack{\beta\in\mathbb{Z}^d \\ \gamma\in\mathbb{Z}^d,|\gamma|>l}} \|\chi_{0,\gamma+\beta}u\|_{L^p}\exp(-|y|\,|\beta|/C_1)\right)dx\,dy \end{aligned}$$

Proof:

We use without explicit reference the equations collected in the above Remark 4.2 and the Helffer-Sjöstrand formula (19). Let $\mathbb{N} \ni q > d/2$. If we multiply

$$g(H_{\omega,l}) = \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{f}}{\partial\bar{z}}(z)(z-H_{\omega,l})^{-1}(\lambda+H_{\omega,l})^{-q}\,dz\wedge d\bar{z} \tag{33}$$

by the characteristic function χ_0 of Λ_1 we get a trace-class operator and consequently

$$\mathrm{Tr}(\chi_0\, g(H_{\omega,l})\chi_0) = \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{f}}{\partial\bar{z}}(z)\,\mathrm{Tr}(\chi_0\,(z-H_{\omega,l})^{-1}(\lambda+H_{\omega,l})^{-q}\chi_0)\,dz\wedge d\bar{z}\ . \tag{34}$$

The same formula holds with H_ω substituted for $H_{\omega,l}$. To bound the trace of $\chi_0\,(H_{\omega,l}-H_\omega)\chi_0$ in mean we estimate

$$\|\chi_0\,(z-H_{\omega,l})^{-1}(\lambda+H_{\omega,l})^{-q}\chi_0 - \chi_0\,(z-H_\omega)^{-1}(\lambda+H_\omega)^{-q}\chi_0\|_{\mathrm{Tr}} \leq \Sigma_1+\Sigma_2$$

by the two summands

$$\begin{aligned} \Sigma_1 &= \|\chi_0\left((z-H_{\omega,l})^{-1}-(z-H_\omega)^{-1}\right)(\lambda+H_\omega)^{-q}\chi_0\|_{\mathrm{Tr}} \\ &= \left\|\chi_0\left((z-H_{\omega,l})^{-1}\left(\sum_{\gamma\in\mathbb{Z}^d,|\gamma|>l}(\omega_{\tilde{\gamma}}-\omega_\gamma)u(x-\gamma)\right)(z-H_\omega)^{-1}\right)\right. \\ &\quad\left.\times\,(\lambda+H_\omega)^{-q}\chi_0\right\|_{\mathrm{Tr}} \end{aligned}$$

and

$$\begin{aligned}
\Sigma_2 &= \|\chi_0(z-H_{\omega,l})^{-1}\left((\lambda+H_{\omega,l})^{-q}-(\lambda+H_\omega)^{-q}\right)\chi_0\|_{\mathrm{Tr}} \\
&= \Bigg\|\chi_0(z-H_{\omega,l})^{-1}\sum_{m=1}^{q}(\lambda+H_{\omega,l})^{m-q-1}\left(\sum_{\gamma\in\mathbb{Z}^d,|\gamma|>l}(\omega_{\tilde\gamma}-\omega_\gamma)u(x-\gamma)\right) \\
&\quad\times(\lambda+H_\omega)^{-m}\chi_0\Bigg\|_{\mathrm{Tr}},
\end{aligned}$$

where in the last equality we used an iterated resolvent formula. Since $|\omega_{\tilde\gamma}-\omega_\gamma|\le\omega_{\max}$ and by standard bounds for the trace norm $\|\cdot\|_{\mathrm{Tr}}$ we have

$$\begin{aligned}
\Sigma_1 &\le \omega_{\max}\sum_{\beta\in\mathbb{Z}^d}\sum_{\gamma\in\mathbb{Z}^d,|\gamma|>l}\left\|\chi_0(z-H_{\omega,l})^{-1}u(x-\gamma)\chi_{0,\beta}\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \\
&\quad\times\|\chi_{0,\beta}(z-H_\omega)^{-1}(\lambda+H_\omega)^{-q}\chi_0\|_{\mathrm{Tr}} \\
&\le \frac{C_1}{|y|^2}\sum_{\beta\in\mathbb{Z}^d}\sum_{\gamma\in\mathbb{Z}^d,|\gamma|>l}\|\chi_{0,\gamma+\beta}u\|_{L^p}\exp(-|y|\,|\beta|/C_1)
\end{aligned}$$

As Σ_2 can be bounded in the same way, our lemma is proved.

q.e.d.

Up to now we followed the proof of Theorem 5.1 of [25] almost literally. From now on we need sharper and more explicit estimates because later we will have to take the limit $l\to\infty$ simultaneously with an approximation $g\to\chi_{[0,E]}$. Special care is needed because the parameters E and l are functions of each other.

Lemma 4.4 *If we choose the constant C_2 sufficiently large and C_3 sufficiently small (depending only on d,δ_2,δ_3 and C_1), we have for all y with $0\neq|y|\le 3$:*

$$\sum_{\beta\in\mathbb{Z}^d}\sum_{\substack{\gamma\in\mathbb{Z}^d\\|\gamma|>l}}\|\chi_{0,\gamma+\beta}u\|_{L^p}\exp(-|y|\,|\beta|/C_1)\le C_2\,e^{-C_3|y|l}\,|y|^{-2d}. \tag{35}$$

The proof of this and the following lemma are given in the appendix.

Lemma 4.5 *Let f be in $C^{n+1}([-1/2,1/2])$ and $\tilde f$ its almost analytic extension of order n. There exists a $l_1:=l_1(d,n,C_3)<\infty$ such that we have for all $l\ge l_1$:*

$$\int_{\mathbb{C}}\left|\frac{\partial\tilde f}{\partial\bar z}(x,y)\right||y|^{-2d-2}e^{-C_3|y|l}\,dx\,dy \quad\le\quad 2C_3^{-n+2d+2}|||f|||_{n+1}\,|\operatorname{supp}f|\,l^{-n+2d+1}\ .$$

We have to bound the derivatives of $f:=(\lambda+\cdot)^q g$ in terms of the derivatives of g itself. A simple calculation using Leibniz' formula shows $|||f|||_{n+1}\le C_4\,|||g|||_{n+1}$, where C_4 depends only on n,q and λ.

We collect the estimates of Lemma 4.3, 4.4 and 4.5 and write down the needed inequalities for our difference of integrals with respect to N and $N_{\omega,l}$.

$$
\begin{aligned}
&\left|\mathbb{E}\left(\int g(x)dN_{\omega,l}(x)\right) - \int g(x)dN(x)\right| \\
\leq\ & \frac{1}{2\pi}\int_{\mathbb{C}} dx\,dy\ \frac{C_1}{|y|^2}\left|\frac{\partial\tilde{f}}{\partial\bar{z}}(x,y)\right| \left(\sum_{\beta\in\Gamma}\sum_{\substack{\gamma\in\Gamma,\\|\gamma|>l}} \|\chi_{0,\beta+\gamma}u\|_{L^p}\exp(-|y|\,|\beta|/C_1)\right) \\
\leq\ & \frac{1}{2\pi}\int_{\mathbb{C}} dx\,dy\ \delta_2 C_1\left|\frac{\partial\tilde{f}}{\partial\bar{z}}(x,y)\right| C_2|y|^{-2d-2}\exp(-C_3|y|l) \\
\leq\ & \frac{\delta_2 C_1 C_2}{\pi C_3^{n-2d-2}}\,|\operatorname{supp} f|\ |||f|||_{n+1} l^{-n+2d+1} \\
\leq\ & C_5\ |\operatorname{supp} f|\ |||g|||_{n+1}\, l^{-n+2d+1}
\end{aligned}
$$

if we choose $l \geq l_1$ and set $C_5 := \frac{\delta_2 C_1 C_2 C_4}{\pi C_3^{n-2d-2}}$. This proves Theorem 4.1 with C_5 as the constant on the rightern side.

q.e.d.

The IDS approximation result (Theorem 4.1) gives information about $N_{\omega,l}$ if properties of N are known. Exploiting this fact, we want to show that $N_{\omega,l}$ is "small" in the energy region where N exhibits a Lifshitz tail. To this end take $g \in C_0^{n+1}(\mathbb{R},[0,1])$ with $g(x) = 1$ for all $x \in [0,E]$ and support in $[-E/2, 2E]$. Moreover let g have minimal derivative in the sense of inequality (24). We estimate

$$
\begin{aligned}
\mathbb{E}\left[N_{\omega,l}(E) - N_{\omega,l}(0)\right] &\leq \mathbb{E}\left(\int g\ dN_{\omega,l}\right) \\
&\leq \int g\,dN + \left|\mathbb{E}\left(\int g\ dN_{\omega,l}\right) - \int g\ dN\right| . \qquad (36)
\end{aligned}
$$

Let $E/2$ be smaller than the gap width b' below the spectral band edge 0. Since $\operatorname{supp} N = \sigma(H_\omega)$ a.s. (c.f.[33]) it follows for $l \geq l_1$

$$\mathbb{E}\left[N_{\omega,l}(E) - N_{\omega,l}(0)\right] \leq N(2E) - N(0)\ +\ C_6\,E^{-n}\,l^{-n+2d+1}\ , \qquad (37)$$

where we used Theorem 4.1 and equation (24). If N has Lifshitz asymptotics at the lower band edge 0, as defined in equation (18), there exists an energy value E_1 such that

$$N(E) - N(0) \leq \exp(-E^{-d/4})\ \ \forall E \in [0, E_1]\ . \qquad (38)$$

Together with (37) this gives

$$\mathbb{E}\left[N_{\omega,l}(E) - N_{\omega,l}(0)\right] \leq e^{-(2E)^{-d/4}}\ +\ C_6\,E^{-n}\,l^{-n+2d+1}\ \forall E \in [0, E_1/2]\ . \qquad (39)$$

For $\alpha \in]0,1[$ we set $E := 2l^{-\alpha}$. This implies

$$\begin{aligned}\mathbb{E}\left[N_{\omega,l}(E) - N_{\omega,l}(0)\right] &\leq \exp(-(4l^{-\alpha})^{-d/4}) + C_6\,(2l^{-\alpha})^{-n}\,l^{-n+2d+1} \\ &= \exp(-4^{-d/4}\,l^{\alpha d/4}) + C_6 2^{\alpha n}\,l^{-n(1-\alpha)+2d+1} \\ &\leq C_6 2^n\,l^{-n(1-\alpha)+2d+1} \qquad (40)\end{aligned}$$

if $l \geq l_2 := l_2(d, n, \alpha, C_6, b', E_1)$. Thus we have proven that the Lifshitz tail of N implies a similar asymptotic behaviour of the IDS of the periodic approximation $H_{\omega,l}$ as stated in the following

Theorem 4.6 *Let N and $N_{\omega,l}$ be the IDS of H_ω and $H_{\omega,l}$ respectively, $n \in \mathbb{N}$ and $\alpha \in]0,1[$. If N has a Lifshitz tail at the lower band edge 0, there exist a $C_7 < \infty$ such that*

$$\mathbb{E}\left[N_{\omega,l}(2l^{-\alpha}) - N_{\omega,l}(0)\right] \leq C_7\,l^{-n(1-\alpha)+2d+1} \qquad (41)$$

for sufficiently large l.

5 Sparsity of states near the lower band edge

We want to estimate the probability of finding an eigenvalue of $H_{\omega,l}(\theta)$ in a small energy interval $I \ni 0$, assuming that N exhibits a Lifshitz tail at 0. Here $H_{\omega,l}(\theta) := H_{\omega,l}|^{\theta}_{\Lambda_{2l+1}} = H_\omega|^{\theta}_{\Lambda_{2l+1}}$ is the operator $H_{\omega,l}$ restricted to $L^2(\Lambda_{2l+1})$ with θ-boundary conditions. The following lemma allows to bound this probability using the IDS of $H_{\omega,l}$.

Lemma 5.1

$$\int_{\theta\in B_l} d\theta\, \mathbb{P}(\{\omega|\ \sigma(H_{\omega,l}(\theta)) \cap [0,E[\neq \emptyset\}) \leq (2\pi)^d\ \mathbb{E}\left(N_{\omega,l}(E) - N_{\omega,l}(0)\right)\ .$$

Proof:

$$\begin{aligned} & \textstyle\int_{\theta\in B_l} d\theta\, \mathbb{P}(\{\omega|\ \sigma(H_{\omega,l}(\theta)) \cap [0,E[\neq \emptyset\}) \\ \leq\ & |\Lambda_{2l+1}|\ \textstyle\int_{\theta\in B_l} d\theta\ \mathbb{E}\left(N(H_{\omega,l}(\theta),E) - N(H_{\omega,l}(\theta),0)\right) \quad \text{Čebyšev inequality} \\ =\ & |\Lambda_{2l+1}|\ \mathbb{E}\,(\textstyle\int_{\theta\in B_l} d\theta\left(N(H_{\omega,l}(\theta),E) - N(H_{\omega,l}(\theta),0)\right) \quad \text{Fubini's theorem} \\ =\ & (2\pi)^d\ \mathbb{E}\left(N_{\omega,l}(E) - N_{\omega,l}(0)\right) \quad \text{equations (15,26)} \end{aligned}$$

q.e.d.

Since the MSA works with specific boundary conditions, e.g. periodic ones, we have to get rid of the average over $\theta \in B_l$ in the last bound. This is possible using the Lipschitz-continuity in θ of the eigenvalues of $H_{\omega,l}(\theta)$.

Lemma 5.2 *For any fixed $\theta_0 \in B_l$ and $E < 1$ we have*

$$\mathbb{P}(\{\omega|\, \sigma(H_{\omega,l}(\theta_0)) \cap [0,E[\neq \emptyset\}) \leq \frac{(2\pi)^d}{|B_l|}\, \mathbb{E}\left(N_{\omega,l}(E + C_9 l^{-1}) - N_{\omega,l}(0)\right) . \quad (42)$$

Proof:

The eigenvalues of $H_{\omega,l}(\theta)$ are Lipschitz continuous in θ, so we have :

$$|E_j(H_{\omega,l}(\theta)) - E_j(H_{\omega,l}(\theta'))| \leq \Xi_{j,l}|\theta - \theta'|$$

for some $\Xi_{j,l} > 0$. One can choose the $\Xi_{j,l}$ independent of j and l only as a function of $E_j(H_{\omega,l}(\theta))$. As we consider only eigenvalues in the energy interval $[0,E[\subset [0,1[$ even this dependence can be eliminated. Thus we can find $\Xi > 0$ such that

$$\Xi \geq \Xi_{j,l} \;\forall l, j \;.$$

Now we can estimate :

$$\begin{aligned}\mathbb{P}(\{\omega|\; \sigma(H_{\omega,l}(\theta_0)) \cap [0,E[\neq \emptyset\}) &= \mathbb{P}(\{\omega|\; \exists j \in \mathbb{N}:\; E_j(H_{\omega,l}(\theta_0)) \in [0,E[\;\})\\ &= \int_{\theta\in B_l} \frac{d\theta}{|B_l|}\, \mathbb{P}(\{\omega|\; \exists j \in \mathbb{N}:\; E_j(H_{\omega,l}(\theta_0)) \in [0,E[\;\}) \quad (43)\end{aligned}$$

If $E_j(H_{\omega,l}(\theta_0)) \in [0,E[$ then $E_j(H_{\omega,l}(\theta)) \in [0, E + \Xi\,\mathrm{diam}(B_l)[\;\; \forall\theta \in B_l$. Using $\mathrm{diam}(B_l) \leq C_8\; l^{-1}$ we bound (43) by

$$\begin{aligned}&\int_{\theta\in B_l} \frac{d\theta}{|B_l|}\, \mathbb{P}(\{\omega|\; \exists j \in \mathbb{N}:\; E_j(H_{\omega,l}(\theta)) \in [0, E + C_9 l^{-1}[\;\})\\ &= \int_{\theta\in B_l} \frac{d\theta}{|B_l|}\, \mathbb{P}(\{\omega|\; \sigma(H_{\omega,l}(\theta)) \cap [0, E + C_9 l^{-1}[\neq \emptyset\;\})\\ &\leq (2\pi)^d\; |B_l|^{-1}\; \mathbb{E}\left(N_{\omega,l}(E + C_9 l^{-1}) - N_{\omega,l}(0)\right)\end{aligned}$$

q.e.d.

We choose now $0 < \alpha < 1$ and $E := l^{-\alpha}$ similarly as before. Thus for $l \geq l_3$ the bound $E + C_9\, l^{-1} \leq 2l^{-\alpha}$ is valid, with l_3 depending on α and C_9. As the IDS is monotone increasing in the energy, this implies

$$N_{\omega,l}(E + C_9\, l^{-1}) \leq N_{\omega,l}(2l^{-\alpha}) \;.$$

If N has Lifshitz tails, we estimate as in Theorem 4.6:

$$\mathbb{E}\left(N_{\omega,l}(2l^{-\alpha}) - N_{\omega,l}(0)\right) \leq C_7\, l^{-n(1-\alpha)+2d+1}$$

for $l \geq l_2$. In this way we obtain from Lemma 5.2

$$\mathbb{P}(\{\omega|\, \sigma(H_{\omega,l}(\theta_0)) \cap [0, l^{-\alpha}[\neq \emptyset\}) \leq C_{10} l^{-n(1-\alpha)+3d+1} \quad (44)$$

since $|B_l|^{-1} \leq const\, l^d$ where the constant depends only on the dimension. The probability in (44) can be bounded by l^{-q} for arbitrary $q > 0$ if

$$\begin{aligned}-\,n(1-\alpha) + 3d + 1 &< -q\\ \Longleftrightarrow n(1-\alpha) &> q + 3d + 1 \quad (45)\end{aligned}$$

and $l \geq l_4 := l_4(d,n,\alpha,q,C_{10})$ is sufficiently large. It is obvious that for any $0 < \alpha < 1$ we can choose n in such a way that the relation (45) is valid.

Similarly, for any fixed $n > q + 3d + 1$ it is possible to choose α sufficiently small, so that (45) holds. Particularly we can choose α from $]0, 1/4[$.

Recall that if H_0 has regular Floquet eigenvalues at the lower spectral band edge 0, the IDS N of $H_\omega := H_0 + V_\omega$ exhibits Lifshitz asymptotics at 0. Thus we proved Proposition 1.2 with $l_0 := \max_{i=1}^4 l_i$.

6 Appendix

Proof of Lemma 4.4. By comparing the Euclidean and sup-norm, the sum in (35) can be bounded by a constant times the integral

$$\int_{\mathbb{R}^d} dx \int_{\|\xi\|_2 > l} d\xi \, e^{-\delta_3 \kappa \|x+\xi\|_2} e^{-|y|\kappa\|x\|_2/C_1}, \qquad \kappa := d^{-1/2}. \tag{46}$$

Substituting $x' = (|y|\delta_3\kappa/6C_1)(2x+\xi), \xi' = (|y|\delta_3\kappa/6C_1)\xi$, using the parallelogram identity for $\|\cdot\|_2$ and $|y| \leq 3, C_1 \geq 1$ we estimate (46) by

$$2^d \left(\frac{3C_1}{\delta_3\kappa|y|}\right)^{2d} \int\limits_{\mathbb{R}^d} dx' \int\limits_{\|\xi'\|_2 > \delta_3\kappa|y|l/6C_1} d\xi' \, e^{-\|x'\|_2 - \|\xi'\|_2} \leq const\, |y|^{-2d} \exp\left(-\frac{\delta_3\kappa}{12C_1}|y|l\right)$$

where the constant depends only on d, δ_3 and C_1.

q.e.d.

Proof of Lemma 4.5. We use inequality (23) and consider first the term:

$$\int_{\mathbb{C}} dx\, dy |y|^{-2d-2} \exp(-C_3|y|l) \frac{3}{\langle x \rangle} \chi_{\{\langle x\rangle < |y| < 2\langle x \rangle\}} \sum_{r=0}^{n} |f^{(r)}(x)| \frac{|y|^r}{r!} \;. \tag{47}$$

The properties of $\langle x \rangle$, s and f ensure $\langle x \rangle \geq 1, 1 < |y| < 3$, thus

$$(47) \leq 6 \int_{\operatorname{supp} f} dx \int_{[1,3]} dy \; e^{-C_3 l} \sum_{r=0}^{n} |f^{(r)}| \frac{3^r}{r!} \leq 60 \; |\operatorname{supp} f| \; |||f|||_n \, e^{-C_3 l}.$$

Now we turn our attention to the other summand in (23)

$$\int_{\operatorname{supp} f} dx \int dy \; |y|^{n-2d-2} e^{-C_3|y|l} \, \frac{|f^{(n+1)}(x)|}{2n!} \\ \leq C_3^{-n+2d+2} |||f|||_{n+1} \; |\operatorname{supp} f| \; l^{-n+2d+1} \;. \tag{48}$$

For sufficiently large l, i.e. $l \geq l'_1(d,n,C_3)$, we have

$$(47) + (48) \leq 2C_3^{-n+2d+2} |||f|||_{n+1} \, |\operatorname{supp} f| \, l^{-n+2d+1}.$$

q.e.d.

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